A Primer of Logic

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PREFACE

In the pages which follow will be found the outlines of the logic of a set of categorical forms which are not Aristotle's own. It is only the fragment of a general theory, but the content of all the chapters of the old logic, which are commonly regarded as essential, except that one which deals with the calculus of classes, will be found included. In the first appendix the relation of these new forms to the traditional ones has been pointed out in detail.

I have at least one debt, which calls for a definite acknowledgement. It is to my friend and teacher, Mr. Edgar A. Singer, Jr., that I owe what training I have had in the science. He has never failed, through hints thrown out in conversation, to correct my misapprehensions. But my indebtedness is more specific than this. I have so far employed his own method, a method developed in his academic lectures, that I could scarcely have ventured upon the publication of these outlines of a theory without his express permission.

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CHAPTER I

§1. In 1846 Sir William Hamilton published the prospectus of an essay on a "New Analytic of Logical Forms,"* which revived the question as to whether or not the quantity of the predicate of the categorical forms should be stated explicitly. The chief difficulties of his system result from the ambiguity of the meaning of *some*, from the impossibility of making every form of categorical expression simply convertible, and from the seemingly curious effort to establish an order of *better* and *worse* between the relations connecting subject and predicate. Four of Hamilton's eight forms are redundant. The four that remain will be represented here by the letters, α, β, γ, ϵ.

Accordingly let

α_{ab} = all a is all b,
β_{ab} = some a is some b,
γ_{ab} = all a is some b,
ϵ_{ab} = no a is b.

Here *some*, the *some* expressed explicitly in β and γ, means *some at least, not all*. This meaning of the word is established unambiguously by the properties of the forms. In addition to these abbreviations we will employ the notation:

x_{ab} = x_{ab} (is true),
x'_{ab} = x_{ab} (is false),
$x_{ab} \cdot y_{ab}$ = x_{ab} (is true) and y_{ab} (is true),
$x_{ab} + y_{ab}$ = x_{ab} (is true) or y_{ab} (is true),
$x_{ab} \angle y_{ab}$ = x_{ab} (is true) implies y_{ab} (is true),
$(x_{ab} \angle y_{ab})'$ = x_{ab} (is true) does not imply y_{ab} (is true).

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*Lectures on Logic, ed. by Mansel and Veitch, Boston, Gould and Lincoln, 1863.

§2. The implications, which are given below, express the chief characteristics of the forms. The theorems follow by the principle of the denial of the consequent, which may be written in the two forms:

$$(x \angle y') \angle (y \angle x') \text{ and } (x' \angle y)' \angle (y' \angle x)'.$$

*Postulates:**

$\alpha_{ab} \angle \beta'_{ab}$	$\beta_{ab} \angle \gamma'_{ab}$	$(\alpha'_{ab} \angle \beta_{ab})'$	$(\beta'_{ab} \angle \gamma_{ab})'$
$\alpha_{ab} \angle \gamma'_{ab}$	$\beta_{ab} \angle \epsilon'_{ab}$	$(\alpha'_{ab} \angle \gamma_{ab})'$	$(\beta'_{ab} \angle \epsilon_{ab})'$
$\alpha_{ab} \angle \epsilon'_{ab}$	$\gamma_{ab} \angle \epsilon'_{ab}$	$(\alpha'_{ab} \angle \epsilon_{ab})'$	$(\gamma'_{ab} \angle \epsilon_{ab})'$
$\gamma_{ab} \angle \gamma'_{ba}$		$(\gamma'_{ab} \angle \gamma_{ba})'$	

Theorems:

$\epsilon_{ab} \angle \alpha'_{ab}$	$\gamma_{ab} \angle \alpha'_{ab}$	$(\epsilon'_{ab} \angle \alpha_{ab})'$	$(\gamma'_{ab} \angle \alpha_{ab})'$
$\epsilon_{ab} \angle \beta'_{ab}$	$\gamma_{ab} \angle \beta'_{ab}$	$(\epsilon'_{ab} \angle \beta_{ab})'$	$(\gamma'_{ab} \angle \beta_{ab})'$
$\epsilon_{ab} \angle \gamma'_{ab}$	$\beta_{ab} \angle \alpha'_{ab}$	$(\epsilon'_{ab} \angle \gamma_{ab})'$	$(\beta'_{ab} \angle \alpha_{ab})'$

Let us postulate in addition that:

$\alpha_{ab} \angle (\alpha'_{ab})'$	$(\alpha'_{ab})' \angle \alpha_{ab}$
$\beta_{ab} \angle (\beta'_{ab})'$	$(\beta'_{ab})' \angle \beta_{ab}$
$\gamma_{ab} \angle (\gamma'_{ab})'$	$(\gamma'_{ab})' \angle \gamma_{ab}$
$\epsilon_{ab} \angle (\epsilon'_{ab})'$	$(\epsilon'_{ab})' \angle \epsilon_{ab}$

Then, if $k_{ab} \mp w_{ab}$ and if k_{ab} and w_{ab} represent only the unprimed letters, α_{ab}, β_{ab}, γ_{ab}, ϵ_{ab}, a complete induction of the propositions given above yields the general result:

$$k_{ab} \angle (k'_{ab})', \quad (k'_{ab})' \angle k_{ab}, \qquad \text{I}$$

$$k_{ab} \angle w'_{ab}, \quad (w'_{ab} \angle k_{ab})', \qquad \text{II}$$

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*These assumptions are in accord with those of the common logic, but no longer hold when the terms are allowed to take on the limiting values 0 and 1; for γ_{01} and ϵ_{01} are both true propositions. The assumptions $(\gamma_{ab} \angle \epsilon'_{ab})'$ and $(\epsilon_{ab} \angle \gamma'_{ab})'$ will be characteristic of a more general logic, which will include the classical logic as a special case. (See the concluding remarks of chapter III.)

and by denial of the consequent,

$$(x \angle y) \angle (y' \angle x'),$$
$$(x \angle y)' \angle (y' \angle x')',$$

it follows that:

$$(k'_{ab})' \angle w'_{ab}, \quad (w'_{ab} \angle (k'_{ab})')'. \qquad \text{III}$$

If now we postulate:

$$(\alpha_{ab} \angle \alpha'_{ab})' \qquad (\alpha'_{ab} \angle \alpha_{ab})'$$
$$(\beta_{ab} \angle \beta'_{ab})' \qquad (\beta'_{ab} \angle \beta_{ab})'$$
$$(\gamma_{ab} \angle \gamma'_{ab})' \qquad (\gamma'_{ab} \angle \gamma_{ab})'$$
$$(\epsilon_{ab} \angle \epsilon'_{ab})' \qquad (\epsilon'_{ab} \angle \epsilon_{ab})'$$

and, consequently,

$$(k_{ab} \angle k'_{ab})', \quad (k'_{ab} \angle k_{ab})',$$

it will follow by*

$$(x \angle y)\ (x \angle z)' \angle (y \angle z)',$$
$$(x \angle z)'\ (y \angle z) \angle (x \angle y)',$$

that $(k_{ab} \angle (w'_{ab})')', \quad ((w'_{ab})' \angle k_{ab})'.$ IV

Finally if we assume

$$\alpha_{ab} \angle \alpha_{ba}, \quad \beta_{ab} \angle \beta_{ba}, \quad \gamma_{ab} \angle \gamma_{ab}, \quad \epsilon_{ab} \angle \epsilon_{ba},$$

it will follow, by

$$(x \angle y)\ (y \angle z) \angle (x \angle z) \text{ and } (x \angle y) \angle (y' \angle x'),$$

that

$$k_{ab} \angle k_{ab}, \quad k'_{ab} \angle k'_{ab}. \qquad \text{I}'$$

Definitions.

If $x_{ab} \angle y'_{ab}$ and $y'_{ab} \angle x_{ab}$, x_{ab} is said to be *contradictory* to y_{ab}. By I, k_{ab} is contradictory to k'_{ab} and, by I′, k'_{ab} is contradictory to k_{ab}.

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*These are obtained from $(x \angle y)\ (y \angle z) \angle (x \angle z)$ by $(xy \angle z) \angle (xz' \angle y')$ and $xy \angle yx$.

If $x_{ab} \angle y'_{ab}$ and $(y'_{ab} \angle x_{ab})'$, x_{ab} is said to be *contrary* to y_{ab}. By II, k_{ab} is contrary to w_{ab}, and conversely, since k_{ab} and w_{ab} are interchangeable.

If $(x_{ab} \angle y'_{ab})'$ and $y'_{ab} \angle x_{ab}$, x_{ab} is said to be *subcontrary* to y_{ab}. By III, k'_{ab} and w'_{ab} are subcontrary pairs.

If $(x_{ab} \angle y'_{ab})'$ and $(y'_{ab} \angle x_{ab})'$, x_{ab} is said to be *subalternate* to y_{ab}. By IV, k'_{ab} and w_{ab} are subalternate pairs.

§3. Having classified the categorical forms under these heads, it remains to differentiate them by means of their formal properties. If we assume as valid,

$$\alpha'_{aa} \angle \alpha_{aa}, \quad \epsilon'_{a\bar{a}} \angle \epsilon_{a\bar{a}}, \quad (\alpha_{aa} \angle \alpha'_{aa})', \quad (\epsilon_{a\bar{a}} \angle \epsilon'_{a\bar{a}})',$$

where $\bar{a}$ represents the class contradictory to a, (non-a), the other propositions given below may be derived.*

$$\begin{array}{llll} \alpha'_{aa} \angle \alpha_{aa} & (\alpha_{aa} \angle \alpha'_{aa})' & \alpha_{a\bar{a}} \angle \alpha'_{a\bar{a}} & (\alpha'_{a\bar{a}} \angle \alpha_{a\bar{a}})' \\ \beta_{aa} \angle \beta'_{aa} & (\beta'_{aa} \angle \beta_{aa})' & \beta_{a\bar{a}} \angle \beta'_{a\bar{a}} & (\beta'_{a\bar{a}} \angle \beta_{a\bar{a}})' \\ \gamma_{aa} \angle \gamma'_{aa} & (\gamma'_{aa} \angle \gamma_{aa})' & \gamma_{a\bar{a}} \angle \gamma'_{a\bar{a}} & (\gamma'_{a\bar{a}} \angle \gamma_{a\bar{a}})' \\ \epsilon_{aa} \angle \epsilon'_{aa} & (\epsilon'_{aa} \angle \epsilon_{aa})' & \epsilon'_{a\bar{a}} \angle \epsilon_{a\bar{a}} & (\epsilon_{a\bar{a}} \angle \epsilon'_{a\bar{a}})' \end{array} \qquad \text{V}$$

The results of V, together with the non-convertible character of γ**, are enough to establish the definitions of the four forms.

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*Under the conditions mentioned above, in note, p. 2, we shall have to write $(\gamma_{a\bar{a}} \angle \gamma'_{a\bar{a}})'$. Implications V are an extension of the meaning of implication, made necessary by our having to call α_{aa} and $\epsilon_{a\bar{a}}$ true propositions. (See Boole, Investigation of the Laws of Thought, chap. XI, p. 169.)

**The operation of simple conversion consists in interchanging subject and predicate. By the principle, $(x \angle z)'(y \angle z) \angle (x \angle y)'$, and what has gone before, we have:

$$(\gamma_{ab} \angle \gamma'_{ab})'(\gamma_{ba} \angle \gamma'_{ab}) \angle (\gamma_{ab} \angle \gamma_{ba})'.$$

Definitions.

A form which is the contrary of itself is called a null-form. By V, β_{aa}, γ_{aa}, ϵ_{aa}, $\alpha_{a\bar{a}}$, $\beta_{a\bar{a}}$, $\gamma_{a\bar{a}}$, are null-forms.

A form which is the subcontrary of itself is called a one-form. By V, α_{aa}, $\epsilon_{a\bar{a}}$, are one-forms.

If x_{ab} is unprimed and x_{aa} a one-form then x_{ab} is called an α-form.

If x_{ab} is unprimed and simply convertible and if x_{aa} and $x_{a\bar{a}}$ are null-forms, then x_{ab} is called a β-form.

If x_{ab} is unprimed and not simply convertible, then x_{ab} is called a γ-form.

If x_{ab} is unprimed and $x_{a\bar{a}}$ a one-form, then x_{ab} is called an ϵ-form.

EXERCISES

(1) Assuming $k_{ab} = k_{ab} \cdot k_{ab}$, $k_{ab} \angle w'_{ab}$, show that,

$$\begin{aligned} \alpha_{ab} &\angle \beta'_{ab}\ \gamma'_{ab}\ \epsilon'_{ab}\ \gamma'_{ba} \\ \beta_{ab} &\angle \alpha'_{ab}\ \gamma'_{ab}\ \epsilon'_{ab}\ \gamma'_{ba} \\ \gamma_{ab} &\angle \alpha'_{ab}\ \beta'_{ab}\ \epsilon'_{ab}\ \gamma'_{ba} \\ \epsilon_{ab} &\angle \alpha'_{ab}\ \beta'_{ab}\ \gamma'_{ab}\ \gamma'_{ba} \end{aligned}$$

by the aid of,

$$(x \angle y)\ (y \angle z) \angle (x \angle z),$$
$$(x \angle y) \angle (zx \angle zy).$$

Equality is defined by

$$(x \angle y)\ (y \angle x) \angle (x = y),$$
$$(x = y) \angle (x \angle y)\ (y \angle x).$$

If now

$$\begin{aligned} \beta'_{ab}\ \gamma'_{ab}\ \epsilon'_{ab}\ \gamma'_{ba} &\angle \alpha_{ab} \\ \alpha'_{ab}\ \gamma'_{ab}\ \epsilon'_{ab}\ \gamma'_{ba} &\angle \beta_{ab} \\ \alpha'_{ab}\ \beta'_{ab}\ \epsilon'_{ab}\ \gamma'_{ba} &\angle \gamma_{ab} \\ \alpha'_{ab}\ \beta'_{ab}\ \gamma'_{ab}\ \gamma'_{ba} &\angle \epsilon_{ab} \end{aligned}$$

it will follow that

$$\begin{aligned} \alpha_{ab} &= \beta'_{ab}\ \gamma'_{ab}\ \epsilon'_{ab}\ \gamma'_{ba} & \alpha'_{ab} &= \beta_{ab} + \gamma_{ab} + \epsilon_{ab} + \gamma_{ba} \\ \beta_{ab} &= \alpha'_{ab}\ \gamma'_{ab}\ \epsilon'_{ab}\ \gamma'_{ba} & \beta'_{ab} &= \alpha_{ab} + \gamma_{ab} + \epsilon_{ab} + \gamma_{ba} \\ \gamma_{ab} &= \alpha'_{ab}\ \beta'_{ab}\ \epsilon'_{ab}\ \gamma'_{ba} & \gamma'_{ab} &= \alpha_{ab} + \beta_{ab} + \epsilon_{ab} + \gamma_{ba} \\ \epsilon_{ab} &= \alpha'_{ab}\ \beta'_{ab}\ \gamma'_{ab}\ \gamma'_{ba} & \epsilon'_{ab} &= \alpha_{ab} + \beta_{ab} + \gamma_{ab} + \gamma_{ba} \end{aligned}$$

The second set of equations follows from the first by the principle, that the contradictory of a product is the sum of the contradictories of the separate factors, and by substituting k_{ab} directly for $(k'_{ab})'$.

(2) Show by the method of the last example that

$$\begin{aligned} \alpha_{ab} &= \alpha_{ab}\ \beta'_{ab} = \alpha_{ab}\ \gamma'_{ab} = \alpha_{ab}\ \epsilon'_{ab} \\ \beta_{ab} &= \beta_{ab}\ \alpha'_{ab} = \beta_{ab}\ \gamma'_{ab} = \beta_{ab}\ \epsilon'_{ab} \\ \gamma_{ab} &= \gamma_{ab}\ \alpha'_{ab} = \gamma_{ab}\ \beta'_{ab} = \gamma_{ab}\ \epsilon'_{ab} \\ \epsilon_{ab} &= \epsilon_{ab}\ \alpha'_{ab} = \epsilon_{ab}\ \beta'_{ab} = \epsilon_{ab}\ \gamma'_{ab} \end{aligned}$$

Show too that

$$\alpha_{ab} = \alpha_{ab}\ \beta'_{ab}\ \gamma'_{ab} = \alpha_{ab}\ \beta'_{ab}\ \epsilon'_{ab} = \alpha_{ab}\ \gamma'_{ab}\ \epsilon'_{ab}, \text{ etc., etc.}$$

and that $\alpha_{ab} = \alpha_{ab}\ \beta'_{ab}\ \gamma'_{ab}\ \epsilon'_{ab}$, etc., etc.

Derive the analogues of the first set:

$$\alpha'_{ab} = \alpha'_{ab} + \beta_{ab} = \alpha'_{ab} + \gamma_{ab} = \alpha'_{ab} + \epsilon_{ab}, \text{ etc., etc.}$$

Accordingly, since $k_{ab} = k_{ab} \cdot w'_{ab}$ and $k'_{ab} = k'_{ab} + w_{ab}$, any primed letter is a modulus of multiplication with respect to any unprimed letter not itself and any unprimed lett r is a modulus of addition with respect to any primed letter not itself.

The propositional *zero* is defined by

$$(xy \angle (xy)') \angle (xy \angle 0),$$
$$(xy \angle 0) \angle (xy \angle (xy)'),$$

and the propositional *one* by

$$((xy)' \angle xy) \angle (1 \angle xy),$$
$$(1 \angle xy) \angle ((xy)' \angle xy).$$

From the principles,

$$(x \angle y') \angle (xy \angle (xy)'),$$
$$(xy \angle (xy)') \angle (x \angle y'),$$

it follows that

$$(x \angle y') \angle (xy \angle 0)$$
$$(x \angle 0) \angle (x \angle y')$$

(3) Derive:

$k_{ab} \cdot k'_{ab} \angle (k_{ab} \cdot k'_{ab})'$; $(k_{ab} + k'_{ab})' \angle k_{ab} + k'_{ab}$;
$k_{ab} \cdot w_{ab} \angle (k_{ab} \cdot w_{ab})'$; $((k_{ab} + w_{ab})' \angle k_{ab} + w_{ab})'$;
$(k'_{ab} \cdot w'_{ab} \angle (k'_{ab} \cdot w'_{ab})')'$; $(k'_{ab} + w'_{ab})' \angle k'_{ab} + w'_{ab}$;
$(k_{ab} \cdot w'_{ab} \angle (k_{ab} \cdot w'_{ab})')'$; $((k_{ab} + w'_{ab})' \angle k_{ab} + w'_{ab})'$.

(4) Show that

$$\alpha_{ab}\, \beta_{ab} \angle 0,\ \alpha_{ab}\, \gamma_{ab} \angle 0, \text{ etc.}$$
$$1 \angle \alpha'_{ab} + \beta'_{ab},\ 1 \angle \alpha'_{ab} + \gamma'_{ab}, \text{ etc.}$$

and that

$$\alpha'_{ab}\, \beta'_{ab}\, \gamma'_{ab}\ \epsilon'_{ab}\, \gamma'_{ba} \angle 0$$
$$1 \angle \alpha'_{ab} + \beta'_{ab} + \gamma'_{ab} + \epsilon'_{ab} + \gamma'_{ba}$$

(5) Assuming $x_{ab} = x_{ab} \cdot x_{ab}$, $x_{ab} = x_{ab} + x_{ab}$, show that

$$\alpha'_{ab}\, \beta'_{ab} = \gamma_{ab} + \epsilon_{ab} + \gamma_{ba},\ \alpha'_{ab}\, \gamma'_{ab} = \beta_{ab} + \epsilon_{ab} + \gamma_{ba}, \text{ etc.}$$
$$\alpha_{ab} + \beta_{ab} = \gamma'_{ab}\ \epsilon'_{ab}\, \gamma'_{ba},\ \alpha_{ab} + \gamma_{ab} = \beta'_{ab}\ \epsilon'_{ab}\, \gamma'_{ba}, \text{ etc.}$$

(6) Derive the general result:

$$(k_{a,b} \angle w_{a,b})'.$$

The comma between the terms indicates that the term order is not fixed. Thus $k_{a,b}$ stands for either k_{ab} or k_{ba}.

(7) From the principle,

$$(x \angle z)'(y \angle z) \angle (x \angle y)',$$

and the postulate $(\alpha_{aa} \angle \alpha'_{aa})'$,
derive $(\alpha_{aa} \angle \beta_{aa})'$, $(\alpha_{aa} \angle \gamma_{aa})'$, $(\alpha_{aa} \angle \epsilon_{aa})'$.

(8) From the principles,

$$(x \angle y)(y \angle z) \angle (x \angle z),$$
$$(x' \angle x) \angle (y \angle x),$$
$$(x \angle x') \angle (x \angle y),$$

and the postulate, $\alpha'_{aa} \angle \alpha_{aa}$, show that all propositions of the form $x_{aa} \angle y_{aa}$, except the three cases in the last example, are valid implications, x_{aa} and y_{aa} representing only the unprimed letters.

(9) By the method of the last example, prove that $(\alpha_{aa} \angle \alpha'_{aa})'$ is the only invalid implication of the form $x_{aa} \angle y'_{aa}$.

(10) Derive seven valid implications in each one of the forms $x'_{aa} \angle y_{aa}$ and $x'_{a\bar{a}} \angle y_{a\bar{a}}$ and nine invalid implications of each one of the same forms.

(11) From $(\epsilon_{a\bar{a}} \angle \epsilon'_{a\bar{a}})'$ by the method of example 7 derive $(\epsilon_{a\bar{a}} \angle \alpha_{a\bar{a}})'$, $(\epsilon_{a\bar{a}} \angle \beta_{a\bar{a}})'$, $(\epsilon_{a\bar{a}} \angle \gamma_{a\bar{a}})'$.

(12) From $\epsilon'_{a\bar{a}} \angle \epsilon_{a\bar{a}}$ by the method of example 8 derive thirteen valid implications.

(13) Show that $(\epsilon_{a\bar{a}} \angle \epsilon'_{a\bar{a}})'$ is the only invalid implication of the form $x_{a\bar{a}} \angle y'_{a\bar{a}}$.

(14) Derive the following implications:

$1 \angle \alpha_{aa}$	$\alpha'_{aa} \angle 0$	$1 \angle \alpha'_{a\bar{a}}$	$\alpha_{a\bar{a}} \angle 0$
$1 \angle \beta'_{aa}$	$\beta_{aa} \angle 0$	$1 \angle \beta'_{a\bar{a}}$	$\beta_{a\bar{a}} \angle 0$
$1 \angle \gamma'_{aa}$	$\gamma_{aa} \angle 0$	$1 \angle \gamma'_{a\bar{a}}$	$\gamma_{a\bar{a}} \angle 0$
$1 \angle \epsilon'_{aa}$	$\epsilon_{aa} \angle 0$	$1 \angle \epsilon_{a\bar{a}}$	$\epsilon'_{a\bar{a}} \angle 0$

$(1 \angle \alpha'_{aa})'$	$(\alpha_{aa} \angle 0)'$	$(1 \angle \alpha_{a\bar{a}})'$	$(\alpha'_{a\bar{a}} \angle 0)'$
$(1 \angle \beta_{aa})'$	$(\beta'_{aa} \angle 0)'$	$(1 \angle \beta_{a\bar{a}})'$	$(\beta'_{a\bar{a}} \angle 0)'$
$(1 \angle \gamma_{aa})'$	$(\gamma'_{aa} \angle 0)'$	$(1 \angle \gamma_{a\bar{a}})'$	$(\gamma'_{a\bar{a}} \angle 0)'$
$(1 \angle \epsilon_{aa})'$	$(\epsilon'_{aa} \angle 0)'$	$(1 \angle \epsilon'_{a\bar{a}})'$	$(\epsilon_{a\bar{a}} \angle 0)'$

Some of the postulates in the text (p. 2) of the form, $(k'_{ab} \angle w_{ab})'$, $(k_{ab} \angle k'_{ab})'$, $(k'_{ab} \angle k_{ab})'$, may be established by reducing them to one of the forms, $(k'_{aa} \angle w_{aa})'$, $(k'_{a\bar{a}} \angle w_{a\bar{a}})'$ $(k_{aa} \angle k'_{aa})'$, etc., cases already considered in preceding exercises

(15) Establish the invalidity of

$$
\begin{array}{lll}
\alpha'_{ab} \angle \beta_{ab} & \alpha'_{ab} \angle \gamma_{ab} & \gamma'_{ab} \angle \gamma_{ba} \\
\beta'_{ab} \angle \gamma_{ab} & \beta'_{ab} \angle \epsilon_{ab} & \gamma'_{ab} \angle \epsilon_{ab} \\
\alpha'_{ab} \angle \alpha_{ab} & \beta'_{ab} \angle \beta_{ab} & \gamma'_{ab} \angle \gamma_{ab} \\
\alpha_{ab} \angle \alpha'_{ab} & \epsilon_{ab} \angle \epsilon'_{ab} & \epsilon'_{ab} \angle \epsilon_{ab}
\end{array}
$$

CHAPTER II

§4. At this point in our theory it will be necessary to introduce certain indefinables, which we shall call the distinctions of *better* and *worse*, following a suggestion of Sir William Hamilton's.* For our immediate purpose it will be enough to define *better than* and *worse than* denotatively, establishing an order among the four forms by a simple enumeration. *Better than* and *worse than* are not transitive relations. When we wish to express the rules for the deduction of the moods symbolically, we shall have to invent symbols to represent *worse than* (/), *doubly worse than* (//), and *trebly worse than* (///). This necessity is avoided in the verbal statement of the principles of deduction by the words "in the same degree" (see rule 1 below).

Definitions.—An ϵ-form is worse than an α-, a β-, or a γ-form.

A γ-form is worse than an α- or a β-form.

A β-form is worse than an α-form.

Best. α-- β-- γ-- ϵ *Worst.*

§5. Immediate inference is a form of implication belonging to one of the types:

1. $x_{ab} \angle y_{ab}$. 2. $x_{ab} \angle y_{ba}$.

These differences are known as the first and the second *figures* of immediate inference respectively.

The part to the left of the implication sign is called the *antecedent;* the part to the right is called the *consequent.*

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*Lectures on Logic, Appendix, p. 536.

Since x and y may take on any of the forms, α, β, γ, ϵ, there will be sixteen propositions of each type, obtained from the permutations of the letters two at a time and by taking each letter once with itself. Each one of the sixteen distinct propositions in each one of the two figures is called a *mood* of immediate inference. The rules which follow below, applied to the postulates, will yield all the true and all the false propositions of each type.

Valid Moods.

1. In any valid mood of the first figure make antecedent and consequent worse in the same degree.

2. In any valid mood convert simply in any form but γ.

Postulate: $\alpha_{ab} \angle \alpha_{ab}$. *Theorems:* The other (6) valid moods.

Invalid Moods.

1. In any invalid mood of the first figure make antecedent and consequent worse in the same degree.

2. In any invalid mood of the first figure interchange antecedent and consequent.

3. In any invalid mood convert simply in any form but γ.

*Postulates:**

$$\{\alpha_{ab} \angle \beta_{ab}\}'; \ \{\alpha_{ab} \angle \gamma_{ab}\}'; \ \{\alpha_{ab} \angle \epsilon_{ab}\}'; \ \{\gamma_{ab} \angle \gamma_{ba}\}'.$$

Theorems: The other (21) invalid moods.

§6. We may also formulate rules for the detection of the invalid moods. These are:

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*The mark (′) over the bracket is intended to indicate that the mood is invalid.

1. If the antecedent be worse than the consequent, the mood is invalid.

2. If the antecedent be better than the consequent, the mood is invalid.

Definition.—Distributed terms are those modified, either implicitly or explicitly, by the adjective *all*, i. e. the subject of the α-, γ- and ϵ-form, and the predicate of the α- and ϵ-form.

3. If a term be distributed in the consequent but undistributed in the antecedent, the mood is invalid.

§7. A syllogism is a form of implication belonging to one of the types:

$$1.\quad x_{ba}\, y_{cb} \angle z_{ca} = (x\, y\, z)_1$$
$$2.\quad x_{ab}\, y_{cb} \angle z_{ca} = (x\, y\, z)_2$$
$$3.\quad x_{ba}\, y_{bc} \angle z_{ca} = (x\, y\, z)_3$$
$$4.\quad x_{ab}\, y_{bc} \angle z_{ca} = (x\, y\, z)_4$$

These differences are known as the first, second, third, and fourth *figures* of the syllogism respectively. The two forms conjoined in the antecedent are called the *premises* and the consequent is called the *conclusion*. The predicate of the conclusion is called the *major* term and points out the *major* premise, which by convention is written first, and the subject of the conclusion is called the *minor* term and points out the *minor* premise. The term common to both premises and which does not appear in the conclusion is called the *middle* term.

Since x, y and z may have any one of the values, α, β, γ, ϵ, there will be sixty-four ways in each one of the four figures, called the *moods* of the syllogism, in which $x\, y \angle z$ can be expressed. There will be consequently two hundred and fifty-six cases to consider. Twenty-nine of these are valid implications; the remaining two hundred and twenty-seven are invalid. From the rules and postulates below, all the moods, valid and invalid, may be deduced.

Valid Moods.

1. In any valid mood of the third figure make a like major premise and conclusion worse in the same degree.
2. In any valid mood of the second figure make a like minor premise and conclusion worse in the same degree.
3. In any valid mood convert simply in any form but γ.

Postulates:	*Theorems:*	
$(\alpha\alpha\alpha)_1$	$(\alpha\alpha\alpha)_{2,3,4}$	$(\beta\alpha\beta)_{1,2,3,4}$
$(\gamma\gamma\gamma)_1$	$(\alpha\beta\beta)_{1,2,3,4}$	$(\gamma\alpha\gamma)_{1,3}$
$(\epsilon\gamma\epsilon)_1$	$(\alpha\gamma\gamma)_{1,2}$	$(\epsilon\alpha\epsilon)_{1,2,3,4}$
	$(\alpha\epsilon\epsilon)_{1,2,3,4}$	$(\epsilon\gamma\epsilon)_2$
	$(\gamma\epsilon\epsilon)_{2,4}$	

Invalid Moods.

1. In any invalid mood of the third figure make a like major premise and conclusion better in the same degree.
2. In any invalid mood of the second figure make a like minor premise and conclusion better in the same degree.
3. If the premises and conclusion of an invalid mood in the fourth figure are all alike, make them all worse in the same degree.
4. If the premises and conclusion of an invalid mood are all alike make the conclusion any degree better or any degree worse.

5. If the premises and conclusion of an invalid mood are all unlike, interchange them in any order.

6. In any invalid mood convert simply in any form but γ.

*Postulates:**

$(\alpha\alpha\beta)'_1$	$(\alpha\beta\epsilon)'_1$	$(\beta\gamma\gamma)'_3$	$(\gamma\gamma\gamma)'_2$	$(\epsilon\beta\epsilon)'_1$
$(\alpha\alpha\gamma)'_1$	$(\alpha\gamma\gamma)'_3$	$(\beta\epsilon\gamma)'_1$	$(\gamma\epsilon\gamma)'_2$	$(\epsilon\gamma\gamma)'_3$
$(\alpha\alpha\epsilon)'_1$	$(\alpha\epsilon\gamma)'_1$	$(\gamma\alpha\gamma)'_2$	$(\epsilon\alpha\gamma)'_1$	$(\epsilon\gamma\epsilon)'_3$
$(\alpha\beta\gamma)'_1$	$(\beta\alpha\gamma)'_1$	$(\gamma\beta\gamma)'_2$	$(\epsilon\beta\gamma)'_1$	

Theorems:

The other (208) invalid moods.

§8. As in the case of immediate inference we may formulate rules for the detection of the invalid moods of the syllogism. These are five in number.

1. A mood is invalid if the conclusion differ from the worse premise.

2. A mood is invalid if an α- and a γ- premise be conjoined in the antecedent and the middle term be undistributed in the major premise.

3. A mood is invalid if the middle term be undistributed in both premises.

4. A mood is invalid if a term which is distributed in the conclusion be undistributed in the premise.

5. A mood is invalid if each premise be in the ϵ-form.**

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*The mark (′) over the bracket is intended to indicate that the mood is invalid.

**These rules are, of course, not sufficient to declare $(\gamma\epsilon\epsilon)_{2,4}$ and $(\epsilon\gamma\epsilon)_{1,2}$ invalid, in case we decide to so regard them. See the concluding remarks of chap. III.

EXERCISES

If $x_{a,b} \angle y_{a,b}$ be denoted simply by (xy), (the comma between the terms indicating that the term order and so the figure, is not determined), the array of sixteen propositions may be constructed thus:

$\underline{\underline{\alpha\alpha}}$	$\beta\alpha$	$\gamma\alpha$	$\epsilon\alpha$
$\alpha\beta$	$\underline{\underline{\beta\beta}}$	$\gamma\beta$	$\epsilon\beta$
$\alpha\gamma$	$\beta\gamma$	$\underline{\gamma\gamma}$	$\epsilon\gamma$
$\alpha\epsilon$	$\beta\epsilon$	$\gamma\epsilon$	$\underline{\underline{\epsilon\epsilon}}$

the moods valid in both figures being underlined twice, the one valid only in the first figure being underlined once. Applying the first rule to the postulate, we obtain in succession, $\beta\beta$, $\gamma\gamma$, $\epsilon\epsilon$, in the first figure; and converting simply in the consequent of those valid in the first figure, except $\gamma\gamma$, we obtain $\alpha\alpha$, $\beta\beta$, $\epsilon\epsilon$ in the second figure.

(1) From the rules and postulates for the derivation of the invalid moods deduce the remaining twenty-one invalid moods.

The rules for the immediate detection of the invalid moods are all *necessary* if we can point to at least one example which falls uniquely under each rule. They are *sufficient* if they declare all the invalid moods to be invalid.

(2) Construct the set of propositions of immediate inference and place after each invalid mood the number of a rule which declares it to be invalid.

(3) Make a list of moods which are declared invalid by the first rule and by no other rule, and a list of moods which are declared invalid by the second rule and by no other rule.

(4) Find an invalid mood which is declared invalid by the third rule and by no other rule and prove that it is the only unique illustration of this rule.

For those who approach the study of the syllogism for the first time, it may facilitate manipulation to point out the general effect of conversion in the form of certain rules.

1. Simple conversion in the major premise changes the first figure to the second and conversely, the third figure to the fourth and conversely.

2. Simple conversion in the minor premies, changes the first figure to the third and conversely, the second figure to the fourth and conversely.

3. Simple conversion in the conclusion changes the first figure to the fourth and conversely and leaves the second and third figure unchanged.

It must of course not escape the beginner's notice that the effect of converting simply in the conclusion is to interchange the premises, since the major term then becomes the minor term and the minor term becomes the major term. The conjunctive relation of logic being commutative, the order of the premises is indifferent, but we agree, as a matter of convention, always to write the major premise first.

(5) From $(\epsilon\gamma\epsilon)_1$, and the third rule under the valid moods alone, deduce $(\epsilon\gamma\epsilon)_2$, $(\gamma\epsilon\epsilon)_2$ and $(\gamma\epsilon\epsilon)_4$.

(6) From the rules and postulates deduce the remaining valid moods.

(7) Assuming only the third rule under the valid moods and the rule: in any valid mood of the *first* figure make a like major premise and conclusion worse in the same degree, deduce all the remaining valid moods from $(\alpha\alpha\alpha)_1$, $(\gamma\gamma\gamma)_1$ and $(\alpha\gamma\gamma)_1$.

(8) Assuming only the second and third rules under the valid moods deduce the remaining valid moods from $(\alpha\alpha\alpha)_1$, $(\gamma\gamma\gamma)_1$, $(\gamma\alpha\gamma)_1$ and $(\epsilon\gamma\epsilon)_1$.

(9) From $(\epsilon\beta\epsilon)'_1$ alone deduce seventy-eight other invalid moods.

(10) From $(\alpha\beta\epsilon)'_1$ alone deduce twenty-three other invalid moods.

(11) Deduce the invalid moods in the first figure which have a γ-minor premise.

The rules for the immediate detection of the invalid moods are *sufficient*, if they declare all the moods not already found to be valid to be invalid. They are all *necessary* if we can point to at least one example which falls uniquely under each rule.

(12) Construct the array of the syllogism and place after each invalid mood the number of a rule that declares it to be invalid.

(13) Show that it follows from one of the rules alone that two β-premises do not imply a conclusion.

(14) Prove that there are only two moods which illustrate the second rule uniquely.

(15) Make a list of examples which fall uniquely under each one of the rules.

CHAPTER III

§9. In this third chapter it is proposed to completely define the relationships of *better* and *worse* by deducing all the true and all the false propositions into which these relationships may enter and then to give a complete expression in the language of symbols of the rules for the deduction of the moods of immediate inference and the syllogism.

§10. Let us, first of all, invent symbols to denote *worse than*, *doubly worse than*, and *trebly worse than*, i. e.

x / y = x is worse than y,
x // y = x is doubly worse than y,
x///y = x is trebly worse than y,

and let us add the following:

Definition.—In the propositions, x / y, x // y, and x///y, x is called the *inferior*, y the *superior* form.

Since x and y may take on any of the four forms α, β, γ, ϵ, there will be sixteen possible propositions of each type, x / y, x // y and x///y, obtained by permuting the letters two at a time and by taking each letter once with itself. The following postulates and principles will yield all the valid moods of each type. We have assumed four principles here because the principles for the deduction of the invalid moods may be derived from these four as theorems.

Principles:

i. (x / y) (y // z) ∠ (x///z) iii. (x / z) (y///z) ∠ (y // x)
ii. (x / y) (z // x) ∠ (z///y) iv. (x / z) (y // z) ∠ (y / x)

Postulates: β / α; ϵ / γ; γ // α.

Theorems: γ / β; ϵ // β; ϵ///α.

We may also formulate rules for the derivation of the moods. It will then be necessary to assume one postulate only.

Definition:—In the propositions, x / y, x // y and x///y, the relation connecting x and y is known as the *worse-relation.*

Definition:—*Trebly worse* (///) is worse than *doubly worse* (//) and doubly worse than *worse* (/). *Doubly worse* (//) is worse than *worse* (/).

The rules are:

1. In any valid mood make superior and inferior form one degree better or one degree worse.

2. In any valid mood make inferior form and worse-relation one degree better or one degree worse.

Postulate:	*Theorems:*
β / α.	The other (5) valid moods.

The invalid moods of each type may be derived from the following postulates and principles:

*Principles:**

i. $(x\ /\ y)\ (x///z)' \angle (y\ //\ z)'$
$(x///z)'\ (y\ //\ z) \angle (x\ /\ y)'$

ii. $(x\ /\ y)\ (z///y)' \angle (z\ //\ x)'$
$(z///y)'\ (z\ //\ x) \angle (x\ /\ y)'$

iii. $(x\ /\ z)\ (y\ //\ x)' \angle (y///z)'$
$(y\ //\ x)'\ (y///z) \angle (x\ /\ z)'$

iv. $(x\ /\ z)\ (y\ /\ x)' \angle (y\ //\ z)'$
$(y\ /\ x)'\ (y\ //\ z) \angle (x\ /\ z)'$

---

*These principles follow from those used for the deduction of the valid moods by $(xy \angle z) \angle (xz' \angle y')$.

Postulates: | *Theorems:*

$(\alpha///\epsilon)'$; $(\beta///\epsilon)'$; $(\gamma///\epsilon)'$; $(\epsilon///\epsilon)'$; $(\epsilon///\gamma)'$; $(\epsilon///\beta)'$. — The other (36) invalid moods.

As in the case of the valid moods, rules may be formulated for the derivation of the invalid moods. Here it will be necessary to assume only three postulates. The rules are:

1. In any invalid mood make superior and inferior form one degree better or one degree worse.
2. In any invalid mood make inferior form and worse-relation one degree better or one degree worse.
3. In any invalid mood make superior form three degrees worse.

Postulates: | *Theorems:*

$(\alpha\ /\ \gamma)'$; $(\gamma\ /\ \alpha)'$; $(\epsilon\ /\ \alpha)'$. — The other (39) invalid moods.

§11. Having now completely defined the relationships of *better* and *worse* by deducing all the propositional forms into which these relationships may enter, there remains for this chapter only one other task, which is to deduce symbolically the moods of immediate inference and the syllogism.

§12. From the postulates and principles, which are given below, all the moods, valid and invalid, of immediate inference may be deduced.

Principles:

i. $(y\ /\ x)\ (x_{a\,b} \angle x_{a\,b}) \angle (y_{a\,b} \angle y_{a\,b})$.
iv. $(x \angle y)\ (y \angle z) \angle (x \angle z)$.

Postulates: | *Theorems:*

$\alpha_{ab} \angle \alpha_{ba}$; $\beta_{ab} \angle \beta_{ba}$; $\epsilon_{ab} \angle \epsilon_{ba}$. — The other valid moods.

*Principles:**

ii. $(x \,/\, y)\ (x_{a\,b} \angle x_{a\,b}) \angle (x_{a\,b} \angle y_{a\,b})'$
$(x//y)\ (x_{a\,b} \angle x_{a\,b}) \angle (x_{a\,b} \angle y_{a\,b})'$

iii. $(y \,/\, x)\ (x_{a\,b} \angle x_{a\,b}) \angle (x_{a\,b} \angle y_{a\,b})'$
$(y//x)\ (x_{a\,b} \angle x_{a\,b}) \angle (x_{a\,b} \angle y_{a\,b})'$

v. $(x \angle y)\ \ (x \angle z)' \angle (y \angle z)'$
$(x \angle z)'\ (y \angle z)\ \angle (x \angle y)'$

Postulates:

$(\alpha_{ab} \angle \epsilon_{ab})'$; $(\epsilon_{ab} \angle \alpha_{ab})'$; $(\gamma_{ab} \angle \gamma_{ba})'$.

Theorems:

The other invalid moods.

§13. All the valid and invalid moods of the syllogism may be deduced from the assumptions which follow. The right to convert simply in any form but γ is expressed under v and vi. It will be evident that some of the postulates might have been saved at the expense of introducing new principles, and conversely. The first two principles for the deduction of the invalid moods under iv are theorems from the ones that have gone before under i, by $(xy \angle z) \angle (xz' \angle y')$.

Principles:

i. $(y \ / \ x)\ (x_{ba}\, z_{bc} \angle x_{ca}) \angle (y_{ba}\, z_{bc} \angle y_{ca})$
$(y \ / \ x)\ (z_{ab}\, x_{cb} \angle x_{ca}) \angle (z_{ab}\, y_{cb} \angle y_{ca})$

v. $(xy \angle z)\ (z \angle w) \angle (xy \angle w)$
$(xy \angle z)\ (w \angle x) \angle (wy \angle z)$
$(xy \angle z)\ (w \angle y) \angle (xw \angle z)$

---

*These principles, except the second under iii are really special cases of principles i and iii under the syllogism, obtained from the latter by making b = c, the primed part of the antecedent in iii becoming unprimed in the special case.

Postulates: $(\alpha\alpha\alpha)_1$; $(\gamma\gamma\gamma)_1$; $(\epsilon\gamma\epsilon)_1$.
Theorems: The other (26) valid moods.

Principles:

ii. $(y \,/\, x)\ (z \,/\, y)\ (xy \angle z)' \angle (xz \angle y)'$
$(y \,/\, x)\ (z \,//\, y)\ (xy \angle z)' \angle (xz \angle y)'$
$(y \,//\, x)\ (y \,/\, z)\ (xy \angle z)' \angle (xz \angle y)'$
$(y \,///\, x)\ (y \,/\, z)\ (xy \angle z)' \angle (xz \angle y)'$
$(y \,/\, x)\ (z \,/\, y)\ (xy \angle z)' \angle (zy \angle x)'$
$(y \,/\, x)\ (z \,//\, y)\ (xy \angle z)' \angle (zy \angle x)'$
$(y \,//\, x)\ (y \,/\, z)\ (xy \angle z)' \angle (zy \angle x)'$
$(y \,///\, x)\ (y \,/\, z)\ (xy \angle z)' \angle (zy \angle x)'$.

iii. $(x \,/\, y)\ (x_{a,b}\, x_{b,c} \angle x_{ca})' \angle (x_{a,b}\, x_{b,c} \angle y_{ca})'$
$(x \,//\, y)\ (x_{a,b}\, x_{b,c} \angle x_{ca})' \angle (x_{a,b}\, x_{b,c} \angle y_{ca})'$
$(y \,/\, x)\ (x_{a,b}\, x_{b,c} \angle x_{ca})' \angle (x_{a,b}\, x_{b,c} \angle y_{ca})'$

iv. $(x \,/\, z)\ (y_{ab}\, x_{cb} \angle x_{ca})' \angle (y_{ab}\, z_{cb} \angle z_{ca})'$
$(x \,/\, z)\ (x_{ba}\, y_{bc} \angle x_{ca})' \angle (z_{ba}\, y_{bc} \angle z_{ca})'$
$(y \,/\, x)\ (x_{ab}\, x_{bc} \angle x_{ca})' \angle (y_{ab}\, y_{bc} \angle y_{ca})'$

vi.* $(xy \angle z)'\ (w \angle z) \angle (xy \angle w)'$
$(xy \angle z)'\ (x \angle w) \angle (wy \angle z)'$
$(xy \angle z)'\ (y \angle w) \angle (xw \angle z)'$

Postulates:

$(\alpha\alpha\beta)'_1$ $(\alpha\beta\gamma)'_1$ $(\alpha\epsilon\gamma)'_1$ $(\beta\gamma\gamma)'_3$ $(\gamma\beta\gamma)'_2$ $(\epsilon\alpha\gamma)'_1$ $(\epsilon\gamma\gamma)'_3$
$(\alpha\alpha\gamma)'_1$ $(\alpha\beta\epsilon)'_1$ $(\beta\alpha\gamma)'_1$ $(\beta\epsilon\gamma)'_1$ $(\gamma\gamma\gamma)'_2$ $(\epsilon\beta\gamma)'_1$ $(\epsilon\gamma\epsilon)'_3$
$(\alpha\alpha\epsilon)'_1$ $(\alpha\gamma\gamma)'_3$ $(\beta\beta\epsilon)'_1$ $(\gamma\alpha\gamma)'_2$ $(\gamma\epsilon\gamma)'_2$ $(\epsilon\beta\epsilon)'_1$ $(\epsilon\epsilon\alpha)'_1$

Theorems:

The other (206) invalid moods.

---

*Principles v and vi are of course not independent. The first under v is a variation of transitivity, the third a variation of the second by $xy \angle yx$. Those under vi follow from those under v by $(xy \angle z) \angle (xz' \angle y')$. Principles v under immediate inference follow from transitivity by the same principle.

§14. We have already pointed out, (note p. 2), that the product $\gamma_{a,b}\ \epsilon_{a,b}$ does not vanish in general if we allow the possibility of the limiting values 0 and 1 for the terms. Under these conditions, $(\gamma\epsilon\epsilon)_{2,4}$ and $(\epsilon\gamma\epsilon)_{1,2}$ are not valid moods of the syllogism, for they become $\gamma_{01}\ \epsilon_{0,1} \angle 0$, for $a=c=0$ and $b=1$. A logic, which recognizes these limiting values of the terms, will have to postulate $(\epsilon\gamma\epsilon)'_1$, say, which yields, $(\epsilon\gamma\epsilon)'_2$ and $(\gamma\epsilon\epsilon)'_{2,4}$.

The only change, which we should then have to make in chapters II and III, would be to replace $(\gamma\gamma\gamma)'_2$ among the postulates by $(\epsilon\gamma\epsilon)'_1$, from which $(\gamma\gamma\gamma)'_2$ follows as a theorem, and to subtract $(\epsilon\gamma\epsilon)_{1,2}$ and $(\gamma\epsilon\epsilon)_{2,4}$ from the list of valid moods.

This logic, which might be called non-Aristotelian, or semi-Aristotelian, or imaginary logic, is more general than the ordinary or classical logic and includes the latter as a special case, becoming, in fact, identical with it when the field of its application is narrowed so as to exclude "nothing" and "universe" as limiting values of the terms. One principle, which is true in the special, but not in the general case, is:

$$(y \;/\; x)\ (x_{ba}\ z_{cb} \angle x_{ca}) \angle (y_{ba}\ z_{cb} \angle y_{ca}),$$

and this principle may be regarded as the differentiating character of the two cases. If we had chosen to assume it, instead of the first principle under i, we could have saved the third postulate, but the second principle under iv would not then have followed as a theorem.

The definitions of chapter I, §3, hold for both cases; the only change to be made in implications V in order to make them true in the more general logic, will be to replace $\gamma_{a\bar{a}}=0$ by $\gamma_{a\bar{a}} \neq 0$, and this property has not been made use of in defining the γ-form.

The Aristotelian forms, A, E, I, O, (see Appendix I), will yield only eight valid moods of the syllogism, under the new condition, instead of the twenty-four valid moods commonly recognized. They satisfy all the conditions of maximum simplicity in the special or classical instance—they are the best possible forms to choose for the construction of an Aristotelian logic—but they fail in the general instance, for they then lose their peculiar advantage, that, corresponding to any member of the set there should be another member of the set which represents its contradictory.

EXERCISES

By the aid of the principles,

$$(xy \angle z)\ (z \angle w) \angle (xy \angle w)$$
$$(xy \angle z)\ (w \angle x) \angle (wy \angle z)$$
$$(xy \angle z)\ (w \angle y) \angle (xw \angle z)$$

we are enabled to convert in either premise or the conclusion. The example which follows will illustrate the method.

$$(\gamma_{ba}\, \alpha_{cb} \angle \gamma_{ca})\ (\alpha_{cb} \angle \alpha_{bc}) \angle (\gamma_{ba}\, \alpha_{bc} \angle \gamma_{ca})$$

(1) From $(\epsilon\gamma\epsilon)_1$ derive $(\gamma\epsilon\epsilon)_{2,4}$.

(2) From the principles and the postulates in the text deduce the remaining valid moods.

(3) From the postulates, $(\alpha\alpha\alpha)_1$, $(\gamma\gamma\gamma)_1$, $(\alpha\gamma\gamma)_1$ and the principle (y/x) $(x_{ba}\, z_{cb} \angle x_{ca}) \angle (y_{ba}\, z_{cb} \angle y_{ca})$ deduce the remaining valid moods.

If we identify the subject and predicate of the conclusion in the mood, $(\beta\alpha\beta)_3$, we obtain $\beta_{ba}\, \alpha_{ba} \angle 0$, (chapter I, implications v). By the aid of $(xy \angle 0) \angle (x \angle y')$ it follows that $\alpha_{ab} \angle \beta'_{ab}$.

(4) Deduce as many as possible of the propositions of the form, $x_{ab} \angle y'_{ab}$, (chapter I, p. 2) by identifying subject and predicate in the conclusion of the valid moods of the syllogism.

The principles,

$$(xy \angle z)'\ (w \angle z) \angle (wy \angle w)'$$
$$(xy \angle z)'\ (x \angle w) \angle (wy \angle z)'$$
$$(xy \angle z)'\ (y \angle w) \angle (xw \angle z)'$$

enable us to convert in either premise or the conclusion.

(5) From $(\beta\beta\beta)'_1$, derive the invalidity of this mood in the other figures.

(6) From $(\epsilon\beta\epsilon)'_1$ alone and principles iii, iv and vi deduce seventy other invalid moods.

(7) From $(\alpha\beta\gamma)'_1$ alone and principles ii deduce nineteen other invalid moods.

(8) Deduce the invalid moods in the third figure, whose conclusion is in the γ-form.

(9) Deduce the invalid moods in the fourth figure, whose major premise is in the γ-form.

(10) From $\gamma\gamma\gamma_1$ alone, deduce forty-six valid implications of the form $L_{a,b}\, M_{b,c} \angle N'_{ca}$,—L, M and N representing only the unprimed letters.

(11) Assuming $\epsilon\gamma\epsilon_{1,2}$ and $\gamma\epsilon\epsilon_{2,4}$ to be invalid, show that $\gamma\gamma\gamma_1$ yields only thirteen valid implications of the form given in the last exercise.

(12) Assuming $(\beta\beta\beta')'_1\ (\beta\beta\gamma')'_1\ (\beta\beta\epsilon')'_1\ (\gamma\gamma\beta')'_2\ (\gamma\gamma\beta')'_3\ (\gamma\gamma\gamma')'_1$ $(\gamma\gamma\epsilon')'_2\ (\epsilon\gamma\beta')'_3\ (\epsilon\gamma\epsilon')'_3\ (\epsilon\epsilon\beta')'_1\ (\epsilon\epsilon\epsilon')'_1$
deduce sixty-nine other non-implications of the same form.

Any non-implication of the form, $L_{b,a}\, M_{c,b} \angle N'_{ca}$, which contains an α-form may be proven invalid by identifying terms in the α-form. Thus $\beta_{ba}\,\beta_{bc} \angle \alpha'_{ca}$ reduces to $\beta_{ba} \angle \beta'_{ba}$ for $c=a$; $\gamma_{ba}\,\alpha_{cb} \angle \gamma'_{ca}$ reduces to $\gamma_{ba} \angle \gamma'_{ba}$ for $c=b$, etc.

(13) Establish the invalidity of the thirty-four non-implications of the form $L_{b,a}\, M_{a,b} \angle N'_{ca}$ not accounted for in the preceding exercise.

(14) Show that there are thirty-six, and only thirty-six, distinct valid implications of the form $L_{a,b}\, M_{b,c}\, N_{c,a} \angle 0$,—L, M and N representing only the unprimed letters, α, β, γ, ϵ.

(15) Derive indirectly the $(\gamma\gamma\gamma')'_1$ of exercise 12.

A certain number of the postulates for the derivation of the invalid moods of the syllogism (p. 21) may be shown indirectly to be invalid by reducing them to invalid moods of immediate inference. Thus $(\alpha\alpha\beta)_1$ reduces to $\alpha_{ab} \angle \alpha'_{ab}$ when the terms in the conclusion are identified, and $(\alpha\beta\gamma)_1$ reduces to $\beta_{ca} \angle \gamma_{ca}$ when we identify terms in the major premise and suppress the part α_{aa} (see chap. IV).

(16) Establish the invalidity of

$(\alpha\alpha\gamma)_1$	$(\alpha\alpha\epsilon)_1$	$(\beta\beta\epsilon)_1$
$(\alpha\beta\epsilon)_1$	$(\alpha\epsilon\gamma)_1$	$(\beta\alpha\gamma)_1$
$(\epsilon\alpha\gamma)_1$	$(\alpha\gamma\gamma)_3$	$(\gamma\alpha\gamma)_2$

Most of the other postulates for the deduction of the invalid moods of the syllogism may be reduced by the method of the following example:

Suppose that $(\epsilon\beta\gamma)_{1}$ is valid. Now $(\epsilon\beta\gamma')_{1}$ is valid by a preceding exercise.

$\therefore (\epsilon_{ba}\,\beta_{cb} \angle \gamma_{ca})\ (\epsilon_{ba}\,\beta_{cb} \angle \gamma'_{ca}) \angle (\epsilon_{ba}\,\beta_{cb} \angle 0)$
since $(x \angle y)\ (x \angle y') \angle (x \angle 0)$.

Consequently $\epsilon_{ba}\,\beta_{cb} \angle \epsilon_{ca}$.

If now we postulate $(\epsilon\beta\epsilon)'_{1}$, it follows that $(\epsilon\beta\gamma)'_{1}$.

(17) Establish the invalidity of

$(\beta\gamma\gamma)_3$	$(\gamma\beta\gamma)_2$	$(\gamma\gamma\gamma)_4$
$(\gamma\epsilon\gamma)_2$	$(\epsilon\gamma\gamma)_3$	$(\epsilon\epsilon\alpha)_1$

The postulates $(\beta\epsilon\gamma)'_{1}$ $(\epsilon\beta\epsilon)'_{1}$ and $(\epsilon\gamma\epsilon)'_{3}$ that remain (p. 21) may be reduced by the following method:

$(\beta_{ba}\,\beta_{cb} \angle \epsilon'_{ca})'\ (\beta_{cb} \angle \gamma'_{cb}) \angle (\beta_{ba}\,\gamma'_{cb} \angle \epsilon'_{ca})'$
by a principle under vi and the postulate of a preceding exercise

$$(\beta_{ba}\,\gamma'_{cb} \angle \epsilon'_{ca})' \angle (\beta_{ba}\,\epsilon_{ca} \angle \gamma_{cb})'$$

which yields $(\beta\epsilon\gamma)'_{1}$ by simple conversion in the major premise.

(18) Establish the invalidity of $(\epsilon\beta\epsilon)_{1}$ and $(\epsilon\gamma\epsilon)_{3}$.

Any non-implication of the form $L_{a,b}\ M_{b,c} \angle N_{ca}$, which contains an α-premise, may be reduced to an invalid mood of immediate inference, and so shown to be invalid, by identifying terms in the α-premise. All of the other invalid moods may be derived from the postulates of exercise (12), the forms of immediate implication given in chapter I, the principles iv and vi of chapter III, together with $(xy \angle z) \angle (xz' \angle y')$ and $(xy \angle z) \angle (z'y \angle x')$.

19. From the postulates of exercise (12) deduce all the non-implications of the form $L_{a,b}\ M_{b,c} \angle N_{ca}$, without making use of principles ii and iii of this chapter.

20. Show that there exist no valid implications of the form $L'_{a,b}\ M_{b,c} \angle N_{ca}$ or $L_{a,b}\ M'_{b,c} \angle N_{ca}$ and consequently none of the form $L'_{a,b}\ M'_{b,c} \angle N_{ca}$ or $L'_{a,b}\ M'_{b,c} \angle N'_{ca}$.

CHAPTER IV

§15. The sorites is a form of implication of the general type:*

$$x_1(1,2)\ x_2(2,3)\ x_3(3,4) --- x_{n-1}(\overline{n-1},\ n) \angle x_n(n\ 1),$$

in which the number of terms is greater than three.

Certain valid moods of the sorites can be constructed from chains of valid syllogisms. Thus the chain of syllogisms:

$$\alpha(1,2)\ \alpha(2,3) \angle \alpha(31),$$
$$\alpha(31)\ \alpha(3,4) \angle \alpha(41),$$
$$\alpha(41)\ \alpha(4,5) \angle \alpha(51),$$

will yield a valid sorites, viz:

$\alpha(1,2)\ \alpha(2,3)\ \alpha(3,4)\ \alpha(4,5) \angle \alpha(51)$, for

$\{\alpha(1,2)\ \alpha(2,3) \angle \alpha(31)\} \angle \{\alpha(1,2)\ \alpha(2,3)\ \alpha(3,4) \angle \alpha(31)\ \alpha(3,4)\}$

$\therefore\ \alpha(1,2)\ \alpha(2,3)\ \alpha(3,4) \angle \alpha(41)$, by the second syllogism and the principle of transitivity.

$$\{\alpha(1,2)\ \alpha(2,3)\ \alpha(3,4) \angle \alpha(41)\} \angle$$
$$\{\alpha(1,2)\ \alpha(2,3)\ \alpha(3,4)\ \alpha(4,5) \angle \alpha(41)\ \alpha(4,5)\}$$
$$\therefore\ \alpha(1,2)\ \alpha(2,3)\ \alpha(3,4)\ \alpha(4,5) \angle \alpha(51),\text{ as before.}$$

Consequently in general, if

$$x_1(1,2)\ x_2(2,3) \angle x_3(31)$$
$$x_3(31)\ x_4(3,4) \angle x_5(41)$$
$$x_5(41)\ x_6(4,5) \angle x_7(51)$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$x_{2n-5}(\overline{n-1}\ 1)\ x_{2n-4}(\overline{n-1},\ n) \angle x_{2n-3}(n1)$$

---

*In this chapter it will be more convenient to employ the notation x(ab) for x_{ab} or x(1,2) for $x_{1,2}$. The comma between the terms means that the term order is not settled.

be a chain of valid syllogisms, then

$$x_1(1,2)\ x_2(2,3)\ x_4(3,4) \text{---}\ x_{2n-4}(\overline{n-1},\ n) \angle\ x_{2n-3}(n1)$$

is a valid mood of the sorites. It remains to be proven that the only valid moods that exist can be constructed from chains of valid syllogisms. The proof depends on the following principles.

Principle i.—A valid mood of the sorites, which has one premise of the same form as the conclusion, will remain valid, when as many of the other premises as we desire are put in the α-form.

Principle ii.—A valid mood of the sorites will remain valid, when as many terms have been identified as we desire.

Principle iii.—An α-premise, whose subject and predicate are identical, may be suppressed as a unit multiplier.

Principle iv.—A valid mood of the sorites, none of whose premises has the same form as the conclusion, will remain valid, when as many premises as we desire are put in the α-form.

Theorem i.—There exists no valid mood of the sorites, in which none of the premises has the same form as the conclusion.

For (principle iv) put all the premises after the first in the α-form. Then by identifying terms (principle ii) the mood of the sorites can be reduced (principle iii) to an invalid syllogism of the form:

$$x_1(1,2)\ \alpha(2,3) \angle\ x_n(31).$$

Conclusion in the α-form.

At least one of the premises is in the α-form (theorem i). If one of the remaining premises, $x_r(\overline{s-1},\ s)$, be not in the α-form, put each one of the other premises in the α-form, if all but x_r be not already in that form (principle i). Then by identifying terms (principle ii) the mood of the sorites will reduce (principle iii) to an invalid syllogism of the form:

$$x_r(\overline{s-1}, s)\ \alpha(s, \overline{s+1}) \angle\ \alpha(\overline{s+1}\ \overline{s-1}),$$
$$\text{or}\ \alpha(\overline{s-2}, \overline{s-1})\ x_r(\overline{s-1}, s) \angle\ \alpha(s\ \overline{s-2}).$$

Consequently all the premises are in the α-form if the mood of the sorites is valid and the sorites is of the general type:

$$\alpha(1,2)\ \alpha(2,3) \text{---}\ \alpha(\overline{n-1}, n) \angle\ \alpha(n1),$$

which can be constructed from the chain of valid syllogisms:

$$\alpha(2,1)\ \alpha(3,2) \angle\ \alpha(31),$$
$$\alpha(31)\ \alpha(4,3) \angle\ \alpha(41),$$
$$\alpha(41)\ \alpha(5,4) \angle\ \alpha(51),$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$\alpha(\overline{n-1}\ 1)\ \alpha(n, \overline{n-1}) \angle\ \alpha(n\ 1).$$

Conclusion in the β-form.

At least one premise, x_t, is in the β-form (theorem i), and all the other premises are in the α-form. For suppose one of the other premises $x_r(\overline{s-1}, s)$ were not in the α-form. Put all the premises (principle i) except x_t and x_r in the α-form. Then by identifying terms (principle ii) the mood of the sorites will be reducible to an invalid syllogism (principle iii) of the form:

$$\beta(\overline{s-1}, \overline{s-2})\ x_r(s, \overline{s-1}) \angle\ \beta(s\ \overline{s-2}),$$
$$\text{or}\ x_r(s, \overline{s-1})\ \beta(s, \overline{s+1}) \angle\ \beta(\overline{s+1}\ \ \overline{s-1}).$$

Consequently the sorites must be of the form:

$$\alpha(1,2)\ \alpha(2,3) \text{---}\ \alpha(s, \overline{s-1})\ \beta(\overline{s+1}, s)\ \alpha(\overline{s+1}, \overline{s+2}) \text{---} \alpha(\overline{n-1}, n)$$

$\angle\ \beta(n\ 1)$, which can be constructed from the chain of syllogisms:

$$\alpha(1,2)\ \alpha(2,3) \angle\ \alpha(31)$$
$$\alpha(31)\ \alpha(3,4) \angle\ \alpha(41)$$
$$\cdots$$
$$\alpha(\overline{s-1}\ 1)\ \alpha(\overline{s-1}, s) \angle\ \alpha(s\ 1)$$
$$\alpha(s\ 1)\ \beta(s, \overline{s+1}) \angle\ \beta(\overline{s+1}\ 1)$$
$$\beta(\overline{s+1}\ 1)\ \alpha(\overline{s+1}, \overline{s+2}) \angle\ \beta(\overline{s+2}\ 1)$$
$$\cdots$$
$$\beta(\overline{n-1}\ 1)\ \alpha(\overline{n-1}, n) \angle\ \beta(n\ 1).$$

Conclusion in the γ-form.

At least one of the premises is in the γ-form (theorem i). Each γ-form in the antecedent must present its terms in the order $(s\ \overline{s-1})$. For suppose that $\gamma(\overline{s-1}\ s)$ should appear as one of the premises. Put each one of the remaining premises in the α-form (principle i). Then by identifying terms (principle ii) the sorites will reduce to an invalid syllogism (principle iii) of the form:

$$\gamma(\overline{s-1}\ s)\ \alpha(s, \overline{s+1}) \angle\ \gamma(\overline{s+1}\ \overline{s-1}),$$

or $$\alpha(\overline{s-1}, \overline{s-2})\ \gamma(\overline{s-1}\ s) \angle\ \gamma(s\ \overline{s-2}).$$

Pursuing the same reasoning as before it can be shown that no β- or ϵ- premises can occur. One form of this sorites may consequently be $\gamma(21)$--$\gamma(n\ \overline{n-1}) \angle\ \gamma(n1)$, which can, in fact, be constructed from the chain of valid syllogisms:

$$\gamma(21)\ \gamma(32) \angle\ \gamma(31)$$
$$\gamma(31)\ \gamma(43) \angle\ \gamma(41)$$
$$\cdots$$
$$\gamma(\overline{n-1}\ 1)\ \gamma(n\ \overline{n-1}) \angle\ \gamma(n\ 1).$$

All the other forms of valid sorites with a γ-conclusion are obtained from the above type by transforming one or more of the premises into the α-form in every possible way under the restrictions of theorem i. Each one of these types can be built up from a chain of valid syllogisms each member of which has one of the forms:

$$(\gamma\gamma\gamma)_{1}, \ (\alpha\gamma\gamma)_{1,2}, \text{ or } (\gamma\alpha\gamma)_{1,3}.$$

Conclusion in the ϵ-form.

At least one of the premises is in the ϵ-form (theorem i), and there is not more than one ϵ-premise. For, if there are two or more ϵ-premises, put all the premises but two of the ϵ-premises in the α-form (principle i). Then by identifying terms (principle ii) we will come upon an invalid syllogism (principle iii) of the form:

$$\epsilon(\overline{s-1},\ s)\ \ \epsilon(s,\ \overline{s+1}) \angle\ \epsilon(\overline{s+1}\ \ \overline{s-1}).$$

There can be present no β-premise. For suppose x_r (s, $\overline{s-1}$) to be a β-premise. Put all the premises except x_r and the ϵ-premise in the α-form (principle i). By identifying terms (principle ii) we will come upon an invalid syllogism (principle iii) of the form:

$$\beta(s,\ \overline{s-1})\ \ \epsilon(s,\ \overline{s+1}) \angle \epsilon(\overline{s+1}\ \overline{s-1}),$$

or $$\epsilon(\overline{s-1},\ \overline{s-2})\ \beta(s,\ \overline{s-1}) \angle\ \epsilon(s\ \overline{s-2}).$$

Any γ-premise coming after the ϵ-premise must present its terms in the order (s $\overline{s-1}$). For suppose γ(s, $\overline{s-1}$) coming after the ϵ-premise to present the term order ($\overline{s-1}$ s). Put all the premises except $\gamma(\overline{s-1}\ s)$ and the ϵ-premise in the α-form (principle i). Then by identifying terms (principle ii) we will come upon an invalid syllogism (principle iii) of the form:

$$\epsilon(\overline{s-1},\ \overline{s-2})\ \gamma(\overline{s-1}\ \ s) \angle\ \epsilon(s\ \overline{s-2}).$$

Any γ-premise coming before the ϵ-premise must present its terms in the order ($\overline{s-1}$ s). For suppose γ(s, $\overline{s-1}$)

coming before the ϵ-premise to present the term order $(s\ \overline{s-1})$. Put all the premises except $\gamma(s\ \overline{s-1})$ and the ϵ-premise in the α-form (principle i). Then by identifying terms (principle ii) we will come upon an invalid syllogism (principle iii) of the form:

$$\gamma(s\ \overline{s-1})\ \ \epsilon(s,\ \overline{s+1}) \angle\ \epsilon(\overline{s+1}\ \overline{s-1}).$$

One form of this sorites may be, consequently,

$\gamma(12)\ \ \gamma(23)$---$\gamma(\overline{s-2}\ \overline{s-1})\ \ \epsilon(s,\ \overline{s-1})\ \ \gamma(\overline{s+1}\ s)$---$\gamma(n\ \overline{n-1}) \angle$ $\epsilon(n\ 1)$, which can, in fact, be constructed from the chain of valid syllogisms:

$$\gamma(12)\ \gamma(23) \angle\ \gamma(13)$$
$$\gamma(13)\ \gamma(34) \angle\ \gamma(14)$$
$$\cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot$$
$$\gamma(1\ \overline{s-2})\ \gamma(\overline{s-2}\ \overline{s-1}) \angle\ \gamma(1\ \overline{s-1})$$
$$\gamma(1\ \overline{s-1})\ \ \epsilon(s,\ \overline{s-1}) \angle\ \epsilon(s\ 1)$$
$$\epsilon(s\ 1)\ \gamma(\overline{s+1}\ s) \angle\ \epsilon(\overline{s+1}\ 1)$$
$$\cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot$$
$$\epsilon(\overline{n-1}\ 1)\ \gamma(n\ \overline{n-1}) \angle\ \epsilon(n\ 1)$$

All the other forms of valid sorites with an ϵ-conclusion are obtained from the above type by replacing one or more of the γ-forms by α-forms in every possible way. Each new type can be constructed from a chain of valid syllogisms, each member of which has one of the forms:

$$(\alpha\,\epsilon\,\epsilon)_{1,2,3,4},\ (\epsilon\,\alpha\,\epsilon)_{1,2,3,4},\ (\gamma\,\epsilon\,\epsilon)_{2,4},\ (\epsilon\,\gamma\,\epsilon)_{1,2}.$$

There exist, consequently, no valid moods of the sorites which can not be constructed from chains of valid syllogisms.*

*If $(\gamma\,\epsilon\,\epsilon)_{2,4}$ and $(\epsilon\,\gamma\,\epsilon)_{1,2}$ are to be regarded as invalid moods, (see the concluding remarks of chapter III), then it can be shown at once that no γ-premise can occur when the conclusion is in the ϵ-form. The general form of such a sorites will be,

EXERCISES

(1) Construct a valid sorites from the chain of valid syllogisms:

$$\alpha_{21}\,\gamma_{32} \angle \gamma_{31},$$
$$\gamma_{31}\,\alpha_{43} \angle \gamma_{41},$$
$$\gamma_{41}\,\gamma_{54} \angle \gamma_{51}.$$

(2) By the aid of the principles of chapter IV, reduce the valid sorites, $\alpha_{21}\,\gamma_{32}\,\alpha_{43}\,\gamma_{54} \angle \gamma_{51}$, successively to each one of the three valid syllogisms of example 1.

(3) Prove the invalidity of the sorites,

$$\gamma_{21}\,\gamma_{32}\,\gamma_{34}\,\gamma_{54} \angle \gamma_{51}.$$

(4) From what chain of valid syllogisms can the sorites, $\alpha_{1,2}\,\gamma_{23}\,\epsilon_{3,4}\,\gamma_{54}\,\alpha_{6,5} \angle \epsilon_{61}$ be constructed?

(5) If $(\epsilon\gamma\epsilon)_{1,2}$ and $(\gamma\epsilon\epsilon)_{2,4}$ be regarded as invalid moods of the syllogism, (see the concluding remarks of chap. III), prove the invalidity of the sorites,

$$\gamma_{12}\,\gamma_{23}\text{--}\gamma_{s-2\ s-1}\ \epsilon_{s,\ s-1}\ \gamma_{s+1\ s}\text{--}\gamma_{n\ n-1} \angle \epsilon_{n1}$$

$$\alpha(1,2)\,\alpha(2,3)\text{---}\alpha(\overline{s-1},s)\,\epsilon(s,\overline{s+1})\,\alpha(\overline{s+1},\overline{s+2})\text{---}\alpha(\overline{n-1},n) \angle \epsilon(n1)$$

which can be built up from the chain of syllogisms,

$$\alpha(1,2)\alpha(2,3) \angle \alpha(31)$$
$$\alpha(31)\alpha(3,4) \angle \alpha(41)$$
$$\cdots\cdots\cdots$$
$$\alpha(\overline{s-1}\ 1)\alpha(\overline{s-1},s) \angle \alpha(s1)$$
$$\alpha(s1)\,\epsilon(s,\overline{s+1}) \angle \epsilon(\overline{s+1}\ 1)$$
$$\epsilon(\overline{s+1}\ 1)\,\alpha(\overline{s+1},\overline{s+2}) \angle \epsilon(\overline{s+2}\ 1)$$
$$\cdots\cdots\cdots$$
$$\epsilon(\overline{n-1}\ 1)\alpha(\overline{n-1},n) \angle \epsilon(n\ 1)$$

APPENDIX I

On the Simplification of Categorical Expression and the Reduction of the Syllogistic Figures

If a and b represent classes, there are four ways in which they may be related categorically, the one standing for subject, the other for predicate. These four forms of relationship are always represented by the letters, A, E, I, O, i. e.

$$
\begin{aligned}
A_{ab} &= \text{all a is b,}\\
E_{ab} &= \text{no a is b,}\\
I_{ab} &= \text{some a is b,}\\
O_{ab} &= \text{not all a is b.}
\end{aligned}
$$

Historical efforts have been made to reduce the number of these relationships. If symbols be invented to denote *some* a ($\breve{a}$) and not-a (a_1), the last three may be represented by means of the first, for:

$$E_{ab} = A_{ab_1},\ I_{ab} = A_{\breve{a}b},\ O_{ab} = A_{\breve{a}b_1}.$$

But an essential difference is here left undistinguished and the number of necessary forms will not have been reduced by this device. If a new symbol be employed for *all* a ($\bar{a}$) and another for the copula *is* ($\angle$), we shall have:

$$
\begin{aligned}
A_{ab} &= \bar{a} \angle \check{b},\\
E_{ab} &= \bar{a} \angle \check{b}_1,\\
I_{ab} &= \breve{a} \angle \check{b},\\
O_{ab} &= \breve{a} \angle \check{b}_1.
\end{aligned}
$$

The four separate categorical forms have, accordingly, been gotten rid of at the cost of introducing four new undefined symbols, so that no economy of our indefinables has been effected.

It is to be observed that the word *some*, which is implicit or explicit in the meaning of part of each proposition, means *some at least, possibly all.* Another set of propositions, in which *some* is to mean *some at least, not all*, may be used to replace the traditional ones. These other forms are:

α_{ab} = all a is all b,
β_{ab} = some a is some b,
γ_{ab} = all a is some b,
ϵ_{ab} = no a is b.

In addition we shall have to employ

the *hypothetical* form,

$x_{ab} \angle y_{ab} = x_{ab}$ implies y_{ab},
$\{x_{ab} \angle y_{ab}\}' = x_{ab}$ does not imply y_{ab},

the *conjunctive* form,

$x_{ab} \cdot y_{ab} = x_{ab}$ and y_{ab},

the *disjunctive* form,

$x_{ab} + y_{ab} = x_{ab}$ or y_{ab}.

Each member of the set, A, E, I, O, may be expressed in the members of the set, α, β, γ, ϵ, and conversely, so that the two are, in fact, logically equivalent, although each one has certain advantages peculiar to itself.

The members of the second set have this property, that, if one is true, then all the others are false.* We assume, accordingly, the

Postulates: $\alpha_{ab} \angle \beta'_{ab}$ $\beta_{ab} \angle \gamma'_{ab}$ $\gamma_{ab} \angle \gamma'_{ba}$
$\alpha_{ab} \angle \gamma'_{ab}$ $\beta_{ab} \angle \epsilon'_{ab}$
$\alpha_{ab} \angle \epsilon'_{ab}$ $\gamma_{ab} \angle \epsilon'_{ab}$

---

*Provided we exclude the limiting values 0 and 1 for a and b. The ordinary definitions of these limits allow γ_{01} and ϵ_{01} be true together.

from which follow, by the principle of the denial of the consequent, the

Theorems:

$$\epsilon_{ab} \angle \alpha'_{ab} \qquad \gamma_{ab} \angle \alpha'_{ab}$$
$$\epsilon_{ab} \angle \beta'_{ab} \qquad \gamma_{ab} \angle \beta'_{ab}$$
$$\epsilon_{ab} \angle \gamma'_{ab} \qquad \beta_{ab} \angle \alpha'_{ab}$$

Consequently*

$$\begin{array}{lll} \alpha_{ab} \cdot \beta_{ab} = 0 & \beta_{ab} \cdot \gamma_{ab} = 0 & \gamma_{ab} \cdot \gamma_{ba} = 0 \\ \alpha_{ab} \cdot \gamma_{ab} = 0 & \beta_{ab} \cdot \epsilon_{ab} = 0 & \\ \alpha_{ab} \cdot \epsilon_{ab} = 0 & \gamma_{ab} \cdot \epsilon_{ab} = 0 & \end{array} \qquad \text{I}$$

From the *Definitions:***

$$\begin{aligned} A_{ab} &= \alpha_{ab} + \gamma_{ab} \\ E_{ab} &= \epsilon_{ab} \\ I_{ab} &= \alpha_{ab} + \beta_{ab} + \gamma_{ab} + \gamma_{ba} \\ O_{ab} &= \epsilon_{ab} + \beta_{ab} + \gamma_{ba} \end{aligned} \qquad \text{II}$$

we obtain immediately***

$$\begin{aligned} \alpha_{ab} &= A_{ab} \cdot A_{ba} \\ \beta_{ab} &= I_{ab} \cdot O_{ab} \cdot O_{ba} \\ \gamma_{ab} &= A_{ab} \cdot O_{ba} \\ \epsilon_{ab} &= E_{ab} \end{aligned}$$

It is an advantage of the forms of the original set, an advantage which the set, α, β, γ, ϵ, does not possess, that the contradictory of any letter is represented by a single other letter of the set. Suppose that we were to

---

*$\alpha\ \beta = 0$ reads: α(*is true*) *and* β(*is true*) is *impossible.*

**A, E, I, O are simply the sums given in equations II. That they are the traditional Aristotelian forms, is only an accident of the reader's application. Hence equations II are definitions and not postulates.

***Multiplying out the sums in II as if they were ordinary polynomials, applying the results of I, and assuming that α, β and ϵ are simply convertible.

combine this advantage with that of simple convertibility in a new set of forms.

To do this it would seem to be enough to subtract from the meaning of A_{ab} the part γ_{ab}, (equations II), and to add this part to the meaning of O_{ab}.* Our new set of forms becomes:

$$\begin{aligned} \alpha_{1ab} &= \alpha_{ab} \\ \epsilon_{1ab} &= \epsilon_{ab} \\ \iota_{1ab} &= \alpha_{ab} + \beta_{ab} + \gamma_{ab} + \gamma_{ba} \\ o_{1ab} &= \epsilon_{ab} + \beta_{ab} + \gamma_{ab} + \gamma_{ba} \end{aligned} \qquad \text{III}$$

the analogues of the old letters being represented by the corresponding Greek vowels.

From equations I and III, and remembering that the sum of α_{ab}, β_{ab}, γ_{ab}, ϵ_{ab}, γ_{ba} makes up the propositional "universe," the results of the following table, yielding all the moods of immediate inference, will easily be seen to hold.

True	*Implies the truth of only*	*Implies the falsity of only*	*False*	*Implies the truth of only*	*Implies the falsity of only*
α	α, ι	ϵ, o	α	o	α
ϵ	ϵ, o	α, ι	ϵ	ι	ϵ
ι	ι	ϵ	ι	ϵ, o	α, ι
o	o	α	o	α, ι	ϵ, o

An induction of these results shows that $\alpha = o'$, $o = \alpha'$, $\epsilon = \iota'$, $\iota = \epsilon'$, and that, consequently, *contradictory*

*Here would seem to be another instance of the manner in which the language of symbols may free a science from the accidents imposed upon its development by the language of speech. The last two members of the new set have apparently no simple verbal expression.

pairs are α, o and ϵ, ι. Likewise *contraries* are α, ϵ; *subcontraries* are ι, o; *subalterns* are α, ι and ϵ, o.

If we define an *affirmative* form as one that becomes unity when subject and predicate have been identified and a *negative* form as one that becomes unity when subject and predicate have been made contradictory, then it is a result of the following

*Postulates:**	α_{aa}	is a true proposition,
	$\epsilon_{a\bar{a}}$	" " " "
Theorems:	β'_{aa}	" " " "
	γ'_{aa}	" " " "
	ϵ'_{aa}	" " " "
	$\alpha'_{a\bar{a}}$	" " " "
	$\beta'_{a\bar{a}}$	" " " "
	$\gamma'_{a\bar{a}}$	" " " "

and equations III, that α and ι are affirmative and that ϵ and o are negative forms.

If a *distributed* term be one modified by the quantitative adjective *all*, it will be seen that α and ϵ distribute both subject and predicate, while ι and o distribute neither. These results are summarized in the table below, the distributed terms being underlined.

	Affirmative	*Negative*
Universal	$\alpha_{\underline{ab}}$	$\epsilon_{a\underline{b}}$
Particular	ι_{ab}	o_{ab}

*x *is true* is to be represented by $x=1$, x *is false* by $x=0$. (See Boole, Investigation of the Laws of Thought, ch. XI, p. 169). The theorems follow by equations I, and equations III become as a result of them, $\alpha_{aa}=1$, $\epsilon_{aa}=0$, $\iota_{aa}=1$, $o_{aa}=0$, $\alpha_{a\bar{a}}=0$, $\epsilon_{a\bar{a}}=1$, $\iota_{a\bar{a}}=0$, $o_{a\bar{a}}=1$. Employing the usual notation, $\bar{a}=$not-a.

The traditional rules for detecting the invalid moods of the old syllogism, constructed from the set A, E, I, O, hold for the new syllogism, built up out of the forms, α, ϵ, ι, o. These rules are:

1. Two negative premises do not imply a conclusion. *Ex.* $\epsilon_{ba}\ \epsilon_{cb} \angle\ \epsilon_{ca}$.

2. Two affirmative premises do not imply a negative conclusion. *Ex.* $\alpha_{ba}\ \alpha_{cb} \angle\ \epsilon_{ca}$.

3. An affirmative and a negative premise do not imply an affirmative conclusion. *Ex.* $\alpha_{ba}\ \epsilon_{cb} \angle\ \alpha_{ca}$.

4. Two premises, in neither of which the middle term is distributed, do not imply a conclusion. *Ex.* $\iota_{ba}\ \iota_{cb} \angle\ \iota_{ca}$.

5. Two premises, in which a given term occurs undistributed, do not imply a conclusion, in which that same term is distributed. *Ex.* $\alpha_{ba}\ \iota_{cb} \angle\ \alpha_{ca}$.

The valid moods which remain, and which of course are valid in all four figures, since each one of the forms is simply convertible, are twelve in number, viz:

$\alpha\alpha\alpha$	$\alpha\alpha\iota$	$\alpha\epsilon\epsilon$	$\alpha\epsilon o$
$\alpha\iota\iota$	$\alpha o o$	$\epsilon\alpha\epsilon$	$\epsilon\alpha o$
$\iota\alpha\iota$	$o\alpha o$	$\epsilon\iota o$	$\iota\epsilon o$

It has been previously observed, (note p. 2), that equations I hold generally only when the limits 0 and 1 are excluded as possible values of a and b. If these possibilities he included, we shall have to assume:

$\{\gamma_{ab} \angle\ \epsilon'_{ab}\}'$ and $\therefore \{\epsilon_{ab} \angle\ \gamma'_{ab}\}'$, since $\gamma_{01}\ \epsilon_{01} \mp 0$. Equations I then become:

$$\begin{array}{lll} \alpha_{ab} \cdot \beta_{ab} = 0 & \beta_{ab} \cdot \gamma_{ab} = 0 & \\ \alpha_{ab} \cdot \gamma_{ab} = 0 & \beta_{ab} \cdot \epsilon_{ab} = 0 & \text{IV} \\ \alpha_{ab} \cdot \epsilon_{ab} = 0 & \gamma_{ab} \cdot \epsilon_{ab} \mp 0 & \end{array}$$

Under these conditions, the fact which the old logic always took for granted, that E_{ab} is the contradictory of I_{ab}, and A_{ab} the contradictory of O_{ab}, no longer holds true. For, while the sum of each of these two pairs of forms is the propositional universe, their product is not the propositional null, (equations II, IV). In order that $E_{ab} \cdot I_{ab}$ and $A_{ab} \cdot O_{ab}$ shall vanish for all values of the terms, it will be necessary to exclude $\gamma_{ab}\ \epsilon_{ab}$ from the product. A new set of forms, in which part of the meaning of ι_{ab} is subtracted from ι_{ab} and added to ϵ_{ab}, will satisfy this requirement. Let this new set be:

$$\begin{aligned}
\alpha_{2ab} &= \alpha_{ab} \\
\epsilon_{2ab} &= \epsilon_{ab} + \gamma_{ab} + \gamma_{ba} \\
\iota_{2ab} &= \alpha_{ab} + \beta_{ab} \\
o_{2ab} &= \epsilon_{ab} + \beta_{ab} + \gamma_{ab} + \gamma_{ba}
\end{aligned} \qquad \text{V}$$

If α, ϵ, ι, o, be replaced by α_2, ϵ_2, ι_2, o_2, respectively in the table, (p. 38), all the results of such a new tabulation will be seen to hold, (equations IV, V). The same definitions as given before will make α_2 and ι_2 affirmative, ϵ_2 and o_2 negative forms, (equations V, and the postulates and theorems, p. 39), but since ϵ_2 distributes neither subject nor predicate, $\epsilon_2\ \iota_2\ o_2$ and $\iota_2\ \epsilon_2\ o_2$ will not be found among the valid moods of the syllogism, (see p. 40). The same rules (p. 40) for the detection of the invalid moods will hold for the new syllogism, but rule 1 is now redundant, being a corollary of rule 4.

It might perhaps appear that our original symmetry, (that of equations I), which was interrupted by the necessity of allowing γ_{01} to stand as a true proposition, could be saved by assuming that the null class exhausts no part of the universe, i. e. *all of nothing is some of everything*, might be regarded as a false proposition. Now $A_{0a} = 1$, or $\alpha_{0a} + \gamma_{0a} = 1$, is Schröder's definition of the null class, and A_{0a} will be a true proposition for all values of a if γ_{01} be true, whereas, $\gamma_{01} = 0$ involves $A_{01} = 0$. These consequences lead us to the alternatives of either giving up our symmetry, (in equations I), or else of regarding the null class as not essential to our algebra.

It is finally to be noted—what was obvious in the beginning—that, while the members of the set, α, ϵ, ι, o, can be expressed in the members of the set, α, β, γ, ϵ, the latter can not be expressed in the former. Consequently, an essential difference has been lost, and the existence of a completed logic of the new forms would not put aside the necessity of working out the logic of the old.

The attempts of the logician to discover a set of categorical forms, which establish a complete symmetry among the moods and figures of the syllogism, are as old as the science itself. The end would be attained if a new set could be selected so as to satisfy the following conditions:

1. Each form of the set must be simply convertible.
2. Corresponding to any member of the set, there must occur another which represents its contradictory.
3. The new set must yield at least one valid mood of the syllogism.
4. Each member of the new set must be representable in the members of a set already proved necessary and sufficient to express all differences, (the set A, E, I, O, say), and conversely.

If x_{ab} be any categorical form, the simplest functions of x, which are themselves categorical and which are in general simply convertible, are $x_{ab} \cdot x_{ba}$ and $x_{ab} + x_{ba}$. It will be enough, therefore, in order to satisfy condition 1, to assume as a new set of forms either such a sum or such a product of each one of the old forms (A, E, I, O, say).

The equation, $\{x_{ab} \cdot x_{ba}\}' = x'_{ab} + x'_{ba}$, suggests at once what our manner of satisfying condition 2 must be, for since the product, $x_{ab} \cdot x_{ba}$ is the contradictory of the sum $x'_{ab} + x'_{ba}$ and x the contradictory of x′, if $x_{ab} + x_{ba}$ [respectively $x_{ab} \cdot x_{ba}$] be chosen as one of our new forms, $x'_{ab} \cdot x'_{ba}$ [respectively $x'_{ab} + x'_{ba}$] must be chosen as one of the others.

Remembering that

$$E_{ab} = E_{ab} \cdot E_{ba} = E_{ab} + E_{ba},$$

and $\quad I_{ab} = I_{ab} \cdot I_{ba} = I_{ab} + I_{ba},$

and that $\quad E_{ab} = I'_{ab}$ and $I_{ab} = E'_{ab},$

it follows that our choice of a new set of forms is limited to the following two:

$$A_1 = A_{ab} \cdot A_{ba}, \qquad E_1 = E_{ab}, \qquad (1)$$
$$O_1 = O_{ab} + O_{ba}, \qquad I_1 = I_{ab},$$

$$A_2 = A_{ab} + A_{ba}, \qquad E_2 = E_{ab}, \qquad (2)$$
$$O_2 = O_{ab} \cdot O_{ba}, \qquad I_2 = I_{ab}.$$

It will be found, however, that the set (2) yields no valid moods of the syllogism. Consequently, applying condition 3, our choice is seen to be unambiguously restricted to set (1), which yields twelve moods, valid each one in each of the four figures. These will be found to be, (dropping the subscripts):

A A A,	A A I,	A E E,	A E O,
A I I,	A O O,	E A E,	E A O,
I A I,	O A O,	E I O,	I E O.

It will be impossible, however, to satisfy condition 4, since every expression involving the new forms will be simply convertible. Consequently, an essential difference has been left undistinguished, and it will not be possible to substitute the new forms for the old. The new forms are, in fact, identical with α_1, ϵ_1, ι_1, o_1, considered above.

From this latter discussion and from the discussion that has gone before, we conclude, that, if it be necessary to retain in our set of forms at least one that is not simply convertible, it will be impossible to satisfy the condition 2 above, unless the null-and one-class be excluded or defined in some way other than ordinary.

APPENDIX II

Historical Note on De Morgan's New Propositional Forms

In introducing the notion of contradictory terms into logic De Morgan discovered two new propositional forms, which cannot be directly expressed by means of the A, E, I, O relations of traditional logic.* Suppose that we denote these two forms by U and V, i. e.

U (ab) = All not a is b,
V (ab) = Some not a is not b,
A (ab) = All a is b,
E (ab) = No a is b,
I (ab) = Some a is b,
O (ab) = Some a is not b.

U and V are simply convertible, for (if $\bar{a}$ = not-a) $U(ab) = A(\bar{a}b) = A(\bar{b}a) = U(ba)$, (converting in A by the principle of contraposition), and

$V(ab) = I(\bar{a}\bar{b}) = I(\bar{b}\bar{a}) = V(ba)$, (converting simply in I).

V distributes both subject and predicate while U distributes neither, for $V(ab) = O(\bar{a}b) = O(\bar{b}a)$ (converting in O by contraposition) and, since O distributes its predicate, both a and b are distributed terms; similarly $U(ab) = A(\bar{a}b) = A(\bar{b}a)$ (converting in A by contraposition) and, since A does not distribute its predicate, a and b are undistributed terms.

If an affirmative form be defined as one that becomes the subcontrary of itself when the subject and predicate

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*Formal Logic, p. 61.

have been identified, and a negative form as one that becomes the contrary of itself under the same conditions, it will be seen that A, I and V are affirmative, that E, O and U are negative forms.

These results are summarized in the following table, the distributed terms being underlined.

	Affirmative	*Negative*
Universal	A($\underline{a}$b)	E($\underline{a}\underline{b}$)
Particular	I(ab)	O(a$\underline{b}$)
Indefinite	V($\underline{a}\underline{b}$)	U(ab)

Below are tabulated all the forms of immediate implication which hold among the six propositions A, E, I, O, U, V. The (144) implications and non-implications necessary to establish unambiguously the results of the table can be derived from a certain number of postulates and the commonly assumed principles of traditional logic.

True	*Implies falsity of*	*Implies truth of*	*False*	*Implies falsity of*	*Implies truth of*
A	E, O, U.	A, I, V.	A	A.	O.
E	A, I.	E, O.	E	E.	I.
I	E.	I.	I	A, I.	E, O.
O	A.	O.	O	E, O, U.	A, I, V.
U	A, V.	O, U.	U	U.	V.
V	U.	V.	V	A, V.	O, U.

An induction of these results will show that:
Contradictory pairs are: A, O; E, I; U, V;
Contrary pairs are: A, E; A, U;

Subcontrary pairs are: I, O; O, V;
Subalternate pairs are: A, I; A, V; I, V; I, U; E, O; E, U; E, V; U, O.

In order to show how the new forms fit the ancient scheme and as an illustration of method let us solve the array of the syllogism. We first observe (see table) that A weakens ambiguously to I or V; that O strengthens ambiguously to E or U.*

Rules:

1. In any valid mood interchange either premise and the conclusion and replace each by its contradictory.

2. In any valid mood strengthen a premise or weaken a conclusion.

3. In any valid mood convert simply in any form but A or O.

Postulates:

A A A (1st figure) is a valid mood.
A U U (" ") " " " "
E A E (" ") " " " "
U E A (" ") " " " "

From these rules and postulates will follow sixteen valid moods in the 1st figure, twenty in each one of the 2nd and 3rd figures, and twenty-one in the 4th figure.

In order to deduce the invalid moods let us assume the

Rules:

1. In any invalid mood interchange either premise and the conclusion and replace each by its contradictory.

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*If x implies y but y does not imply x, then x is said to be a *strengthened* form of y, and y is said to be a *weakened* form of x.

2. In any invalid mood weaken a premise or strengthen a conclusion.

3. In any invalid mood convert simply in any form but A or O; and

Postulates:

A A A	(4th	figure)	is	an	invalid	mood.
A A O	("	")	"	"	"	"
A A V	(3rd	")	"	"	"	"
A A I	(2nd	")	"	"	"	"
A A O	(1st	")	"	"	"	"
E E I	("	")	"	"	"	"
E E O	("	")	"	"	"	"
E U A	("	")	"	"	"	"
E U O	("	")	"	"	"	"
U U O	("	")	"	"	"	"
U U V	("	")	"	"	"	"

From these postulates and rules follow the remaining (772) invalid moods.

It will be seen at once that the rules of the old logic for the immediate detection of the invalid moods of the syllogism no longer hold. To give only one illustration: A term may appear distributed in the conclusion of a valid mood and be undistributed in the premise.